NAVAL POSTGRADUATE SCHOOL

Monterey, California

A REVENUE AND EXPENSE APPORTIONMENT CONCEPT FOR THE ANALYSIS OF INTERNAL RETURNS ON INVESTMENT (THE SIMPLE CASE)

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This paper presents a technique which uses revenue, expense, and investment data to assign returns on investment to each activity in an interactive economic system. It is a simple case of a more general model. The economic system may be a group of firms, a single firm, or a part of a firm. The technique, referred to here as revenue and expense apportionment, is particularly suited for joint cost and joint product situtations such as those encountered in transportation, petrochemical production, and industrial funded activities in the

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Government. The principal theme of the apportionment concept is that revenues and expenses can be logically distributed among activities by using a transfer pricing mechanism so as to reveal how each segment shares in a system's overall return on investment, even when joint products are involved. The apportionment concept assigns returns to the activities involved in creating products and services; this is in contrast to approaches that distribute revenues and expenses directly to products and services. The apportionment concept is also linked to profit maximization concepts when the model of the system meets some fundamental requirements. The managerial and economic applications of the apportionment methodology include segmental investment evaluation, limited segmental performance analyses, pricing analyses and regulation. Included in this paper is a brief discussion of existing approaches, a description of the way in which a system is to be modelled in order to apply the apportionment technique, a discussion of apportionment concepts, their interpretation, numerical examples, and some economic implications.

Introduction

This paper presents a method for apportioning revenues and expenses among segments of an enterprise so as to reveal how each segment shares in overall return on investment. The results would bring the attention of management to those areas exhibiting irregular returns. Presumably, such segments would be reviewed for appropriate action.

The methodology presented here is a technique for determining internal transfer prices for resources which do <u>not</u> have established market values. Internal prices, or values, are imputed for such resources by evaluating their role in the creation of revenues and expenses among the associated segments of the enterprise. With these imputed prices, along with the prices for resources which have established markets, the "profit" for each segment is computed. The ratio of "profits" to investment in each segment gives rates of return.

Obviously, the central issue here is how a resource is valued when no established market for the resource exists. The approach here is based on the general premise that, if a resource does not have an applicable value external to the enterprise, then it can only be valued in relation to its internal creation and use. Loosely speaking, the internal value of a resource is taken here to be dependent on the "returns" that it ultimately creates less its "costs."

The orientation of the apportionment methodology in this paper is restricted to the analysis of a firm, or a portion thereof. Larger scale applications, such as in a planned economy, are certainly possible but not emphasized here. Also, the retrospective nature of the technique should be apparent. Transfer prices are developed "after the fact," and used for historical evaluation.

Background

The concepts developed in this paper arose from research which was attempting to identify the profitability of freight traffic segments in a motor carrier system. The problem was not readily resolvable because of the joint nature of vehicle movements. When trucks and trailers are continually committed to move cargo from, say, point A to point B, there necessarily persists the potential for moving other cargo in the opposite (backhaul) direction. When cargo is actually moved in the opposite direction, a joint cost allocation problem arises because a portion of the backhaul trucking cost necessarily exists independent of the backhaul cargo.

It has been recognized for a long time that joint costs cannot be rationally allocated.[1] Nevertheless, that does not imply that the joint product problem defies economic analysis. A number of methods deal with the joint cost problem, and the appropriate method for a given situation depends on what information is available, and the objectives of the analyst. Two approaches are discussed below.

If one is attempting to determine how production levels and/or prices ought to be changed so as to improve profits, then marginal cost and marginal revenue analysis, along with some mathematical programming techniques, could be used.[2] Of course, reasonably accurate estimates of marginal effects must be available in order to take this approach.

In another scenario where marginal revenue and marginal cost data are not immediately available, one may be simply attempting to identify the segments in the enterprise which are producing below average returns so that corrective action can be focused on those areas. In this case, intra-company prices (transfer prices) can be attached to goods and services crossing departmental lines so as to show profit performance on a departmental basis.[3] This approach is very effective when transfer prices can be rationally determined from, say, cost accounting data or open market prices. Conversely, if transfer prices cannot be rationally determined, as in the case of joint products with ambiguous open market values, then the approach breaks down.

Returning to the motor carrier problem, it is amenable to marginal analysis when, of course, marginal revenue and cost data are available.[4] The application of transfer prices to the problem was not, at first, fruitful because of certain pricing ambiguities that persisted. (The transfer price concept applies here in the context that a terminal receiving a vehicle would pay a premium to the terminal from which the vehicle originated.) Nevertheless, a unique method was finally discovered which created transfer prices by balancing the returns on investment among the various segments of the motor carrier system. The results were intuitively appealing, and further analysis revealed that it had some very attractive properties which coincidentally related to optimization. It was also realized that this approach applied to general economic systems, especially in instances where joint and/or intermediate products have ambiguous values. In a nutshell, this was how the apportionment concept was developed. The results provide a useful diagnostic tool for identifying how individual segments share in system wide returns on investment.

Perspective

In essence, the apportionment methodology is a transfer pricing procedure which balance returns on investment among producers and consumers of internal resources. More specifically, internal transfer prices are selected which equitably spread returns across producers and consumers of internal resources. This is not saying that internal transfer prices are selected which equalize the returns among the producers of a particular resource, nor is it saying that returns are equalized among the consumers of a particular resource. Rather, it is saying prices are selected such that returns for the internal producers of a particular resource will, on an average, equal the returns of the consumers of the resource.

In other words, prices are developed which do not let the internal users of a particular resource reap any greater, or less, average returns than the internal producers of the resource. Keep in mind that it has already been stated that this approach would be applied only to resources which do not have established external market values. Therefore, this internal valuation concept cannot contradict external valuations because external valuations are not available.

Applying these internal prices to an enterprise reveals how each segment shares in total returns with respect to the structure and returns of adjacent segments. In this sense, apportioned returns are relative, as opposed to absolute, measurements and must be interpreted as such. Also, it is worth emphasizing that the apportionment methodology does not measure returns which are realized upon the sale of a product. Rather, it indicates how, say, raw material support, manufacturing, and physical distribution departments <u>share</u> in the returns brought about by the creation and sales of products.

The General Case

As indicated in the title, the coverage here is limited to a so called simple case. The presumption supporting the simple case is that internally transferred resources have either an established external value, or an entirely unknown external value. In the former case, the resource has an established market value. In the latter case, the resource can neither be bought nor sold externally; the lack of a market does not imply that the resource has zero value, but rather implies that the resource has a completely ambiguous value; the apportionment methodology values such resources according to their internal usage.

The general apportionment model differs from the above in that lower and upper bounds on the values of resources can be taken into account. Instead of requiring resources to have either a precise value or an entirely ambiguous value, the general model allows for situations where a resource has a definite minimum value and/or a maximum value. The internal values of such resources are limited within these ranges.

The characteristics of the general model are conceptually similar to the properties of the simple model. The coverage of the simple model here provides basic insights to the more general case, which is the topic of a forthcoming paper.

Interpretations

The apportionment methodology is a set of presumptions which allow the total returns of an enterprise to be uniquely distributed among its constituent segments. Obviously, apportionment results must be interpreted in terms of these assumptions.

The underlying principle here is that transfer prices are determined for internal resources of ambiguous value so as to "balance" returns among producers and users of these resources. This means that the imputed return of a particular segment is dependent on its own performance along with the performance of segments with which it exchanges resources. Apportionment results do not assess the absolute economic efficiency of each segment, but instead assess the relative economic positions of the segments in the system.

If the apportioned returns of a particular segment are extraordinary, then the implication is that the segment itself and/or adjacent segments are responsible for the abnormality. In this sense, the apportionment methodology only expresses symptoms, not causes. It would bring the attention of management to problem areas. Presumably, corrective action would be developed from additional analyses.

Utility

The apportionment methodology will not produce new information about situations where all internal resources have precise market values. Similarly, it would not likely produce new revelations about situations where there are no joint products, and cost elements can be neatly traced through every segment of the system. The value of the apportionment methodology is realized in systems involved with joint products of ambiguous value. Its utility becomes greater as a system becomes more complex.

The arbitrary assignment of joint costs can be useful, even if not rational, in simple situations. However, as complexity increases, arbitrariness is compounded to the extent that results become impossible to interpret. In contrast, the apportionment methodology is a uniform procedure for distributing returns, and therefore provides a consistent point of reference.

It could be said that the apportionment procedure is itself arbitrary. Nevertheless, it is consistent and therefore certainly less arbitrary than, say, a heterogeneous set of rules applied indiscriminately across the enterprise.

Organization

The next section discusses how a system is to be modelled so as to apply the apportionment methodology. In doing so, it lays the foundation for the conceptual development of the approach taken by the methodology.

After the modelling section, the conceptual principles of apportionment are covered, along with its mathematical characteristics. A fundamental knowledge of matrix algebra is required to grasp the general meaning of these concepts; a thorough knowledge of matrix algebra is required to follow the logic of the proofs.

Interpretations of a few numerical examples are then covered, along with conceptual applications and an evaluation.

Modelling Methodology

This section discusses how an economic system is to be modelled so as to provide a basis for apportionment concepts. After a few general remarks, the discussion will focus on the terminology and structure of the modelling procedure. This will be followed by a discussion of how a model is to be represented both graphically and mathematically. Examples are then illustrated. Lastly, there is a discussion of qualifications which specify certain requirements that should be met when developing a model.

General Remarks

The purpose of a model here is to clearly depict the flow of resources among the segments of the system being examined. The segments, sometimes referred to as activities, are represented as components. The components along with the resource linkages which connect the components, make up the model of the system.[5]

Later on in this section, the focus will be on a sub-set of components in the system. Specifically, only those components involved with ambiguous valued resources will be of concern. So keep in mind that the ensuing general discussions will later be narrowed to include only essential elements.

The specific resource flows and activities to be modelled depend on the nature of the managerial problem which the apportionment methodology is being used to resolve. This is no different than any other analytical tool; for example, the structuring of a linear programming model depends on the problem to which the concept of optimization is being applied. To discuss intelligently which resources and activities should be modelled first requires an understanding of apportionment concepts; therefore, applications are covered in a later section. For the time being, resource and activity structures will be assumed without elaboration, and for the sake of simplicity, only single time period models will be covered.

Terms and Structure

The terminology and building blocks used in modelling a system are presented here. General definitions and assumptions are first stated, and then followed by brief examples.

System Definition

Presumably, the analyst is faced with the task of assessing the returns, on a segmental basis, of a business organization. This provides the foundation for identifying the activities which fall within the scope of the study, and those which do not. The activities under study are called endogenous activities. The activities which provide the external surroundings for the endogenous activities are called exogenous activities. Typically, resources are passed from exogenous activities to endogenous activities, where they are transformed and passed to other endogenous activities, and eventually passed back to exogenous activities.

On a conceptual level, the system is defined to include both exogenous and endogenous activities, along with the resources which flow among them. Nevertheless, the focus is of course on endogenous activities.

System Components and Resources

The model of a system is composed of system components and system nodes. System components represent activities of the system, and system nodes represent the points at which resources (goods and/or services) are transferred among system components. Resources consumed by a system component are called inputs, and those produced are called outputs of the system component. System nodes connect the outputs of system components to inputs of other system components.

Components, for example, might represent the following activities in a firm:

1) the production of a particular product:
2) the purchasing and storage of raw material(s), or the storage and sale of finished products;
3) the sales effort for a particular product;
4) the movement of cargo from one place to another;
5) the providing of equipment maintenance services to several departments.

Similarly, nodes might represent the points where:

1) produced goods are transferred from a production line to storage facilities;
2) raw materials are transferred from storage to production processes;
3) raw materials are transferred from the vendor's possession to the firm's possession;
4) cargo is transferred from storage to a transportation mode for movement;
5) finished goods transferred from the firm's possession to a customer's possession.

Components can obviously be designated as exogenous or endogenous according to the classification of the activity they represent. Exogenous components represent exogenous activities, and endogenous components represent endogenous activities. Applying these definitions to a model of a business firm, endogenous components would represent activities such as procurement, storage, production, and marketing, whereas exogenous components would represent external entities such as suppliers, customers, and sources of labor.

Resources fall into one of three classes. The classifications are exogenous, peripheral, or endogenous. A resource transferred strictly among exogenous activities is an exogenous resource. A resource transferred strictly among endogenous activities is an endogenous resource. A resource transferred among endogenous and exogenous activities is a peripheral resource. Notice that this resource classification scheme is dependent upon the components that create and consume the resource, and not dependent on the physical attributes of the resource.

As previously mentioned, nodes represent the points at which resources are transferred among components. Specifically, a node represents the point where the output resource of a component, or set of components, is transferred as an input resource to another component, or set of components. Obviously, a node can be classified as exogenous, peripheral, or endogenous, depending upon the types of components it connects. An exogenous node only connects exogenous components. An endogenous node only connects endogenous components. A peripheral node connects endogenous components with exogenous components.

Priced and Unpriced Resources

In addition to the classification described in preceding sections, resources and nodes are also designated as being priced or unpriced. Priced resources are actively traded in open markets, and therefore have definite values. Unpriced resources are neither bought nor sold in open markets, and therefore have ambiguous values. (Recall the statements made at the beginning of the paper which emphasize that the concepts here presume all all resources fall into one of these two distinct categories; the general model must be used otherwise.)

For a priced resource, the known assigned price is incorporated in the model by having the components which consume the resource pay the components which produce the resource. As will be defined later, component payments for inputs are called expenses, and receipts from outputs are called revenues.

For initial modelling purposes, unpriced resources are transferred from one component to another without payments going from the recipient to the source. It is the objective of the apportionment methodology to eventually assign imputed transfer prices to these unpriced resources and then apply imputed payments to the endogenous components. In other words, unpriced resources are initially treated as having no value, and then later are assigned imputed prices according to apportionment criteria which take endogenous interrelationships into account. In this sense, unpriced resources are ultimately given <u>relative</u> endogenous values, but for terminology purposes they are still referred to here as being unpriced.

It is important to note that the endogenous set of activities must be such that all peripheral resources are priced resources. This follows from the conceptual contradiction which arises when an unpriced resource is transferred between an exogenous component and an endogenous component; in such a case, one would be trying to impute a transfer price for a resource without looking at the exogenous factors affecting the resource. Therefore, when one is designating endogenous activities, enough activities must be included so that only priced resources are being exchanged between the endogenous set of activities and the exogenous set of activities. This is hardly a problem when modelling a business firm because explicit expenses are realized when procuring resources from external entities, and explicit revenues are realized when selling resources to external entities, and therefore, such resources are necessarily priced.

Revenue, Expenses, and Contribution

Revenues and expenses arise from the consumption and production of *priced* resources by a component during the modelled time interval. A component's *revenue* is the total priced value of priced outputs created by the component during the time interval. A component's *expense* is the total priced value of priced inputs consumed by the component during the time interval. Depreciation and amortization would be necessary for the equitable assignment of capital equipment costs to the single time period expense.

A component's *contribution* is its revenue minus its expense. In many instances a component may have no revenue, in which case contribution is the negative value of its expense.

Investment Levels

The apportionment methodology requires that average investment levels in each component be ascertained. The exact procedure to be used for measuring investment levels in a component would depend on the analyst's view of the system. Therefore, a general procedure will be discussed here, keeping in mind that other approaches could be used. Whatever approach is used, apportionment results would necessarily be interpreted in that light.

Inasmuch as a manager is interested in investment because there are economic opportunities foregone when resources are tied up, the assessment of investment levels in components here will be based upon the estimated *value* of the resources tied up in a component. (This approach is in contrast with those based purely on historical *costs*.)

Values of facilities, equipment, and inventories would be assessed at their individual market value. In most practical cases it would be necessary to use estimated values of facilities and equipment over anticipated life cycles. In cases where inventory levels fluctuate within the time interval being represented, it would be necessary to calculate average levels, and from that, average values.

Apportionment Components and Nodes

The apportionment methodology is concerned here with developing imputed transfer prices for endogenous unpriced resources. In effect, revenues and expenses are redistributed (apportioned) among endogenous components involved with unpriced resources. Because only those components involved with unpriced resources will be affected, it is prudent here to limit our attention only to those endogenous components and endogenous nodes directly involved with unpriced resources.

Without prejudice to the ways in which the terms component and node were used in previous discussions, henceforth, the term component will refer only to those endogenous components involved with one or more unpriced resources. Also, the term node will refer only to those endogenous nodes representing the transfer of unpriced resources; similarly, the terms inputs and outputs will refer only to unpriced resources.

Representation

To facilitate the illustration of a model, a format will be adopted for graphical representations. Also, some mathematical variables will be adopted to represent measures. As stated before, only those components and nodes involved with unpriced inputs and outputs will be represented.

Graphical Representations

A small triangle will be used to represent a component. Lines attached to a vertex of a triangle will represent unpriced outputs of the component; lines attached to a side of a triangle (not at a vertex) will represent unpriced inputs to the component. Arrows on the lines will indicate the direction of the flow of the unpriced input/output. Small circles will represent nodes.

The numbers inside the triangles will enumerate each component. The numbers inside the circles will enumerate each node.

Mathematical Notation

Let λ equal the number of components in the model. As stated earlier, each of these components will have at least one unpriced input and/or output.

Let $a(j)$ equal the average investment level in component j. The index j will run from 1 to λ, $(j=1,\ldots,\lambda)$. Let the λ by 1 column vector $\underset{\cdot}{a}$ represent the investment levels of the components, where the (j)th element in $\underset{\cdot}{a}$ is $a(j)$. The value of $a(j)$ would be in terms of dollars.

Let the variable $r(j)$ equal the revenues created by component j. Let the variable $e(j)$ equal the expenses created by component j. Let $c(j)$ equal the contribution of component j; of course, $c(j)$ equals $r(j)$ minus $e(j)$.

Let the λ by 1 column vectors r, e, and c represent the revenues, expenses, and contributions, respectively, of the components. The (j)th element in each will pertain to the (j)th component. By definition: c equals $r-e$; and $c'\cdot\underline{1}$ equals total contribution of all components. (The apostrophy denotes the transpose of the vector, and $\underline{1}$ is a column vector of ones.)

Let Θ equal the number of nodes. As stated earlier, each node represents a transfer point for an unpriced resource. Each node will be indexed by the letter i, where i runs from 1 to Θ.

Let $P_\beta(i,j)$ equal the proportion of the unpriced resource flowing through node i that is provided by component j. For example, if the total resource flow through node five is 50 units, and 35 of these units come from component nine, then $P_\beta(5,9)$ equals (.7). It should be clear that the following relationships hold:

1) if component j does not have an output connected to node i, then $P_\beta(i,j)$ equals zero;

2) if component j is the only component providing resources to node i, then $P_\beta(i,j)$ equals one;

3) each $P_\beta(i,j)$ is greater than or equal to zero, and less than or equal to one;

4) the sum of $P_\beta(i,1) + P_\beta(i,2) + \ldots + P_\beta(i,\lambda)$ will necessarily equal one for each node.

Let P_β be a matrix consisting of Θ rows and λ columns; the value in the (i)th row and (j)th column will be $P_\beta(i,j)$.

Let $P_\alpha(i,j)$ equal the proportion of the unpriced resource flowing through node i that goes to component j. For example, if the total resource flow through node eight is 20 units, and 5 of these units go to component eleven, then $P_\alpha(8,11)$ equals (.25). It should be clear that the following relationships hold:

1) if component j does not have an input connected to node i, then $P_\alpha(i,j)$ equals zero;

2) if component j is the only component drawing resources from node i, then $P_\alpha(i,j)$ equals one;

3) each $P_\alpha(i,j)$ is greater than or equal to zero, and less than or equal to one;

4) the sum of $P_\alpha(i,1) + P_\alpha(i,2) + \ldots + P_\alpha(i,\lambda)$ will necessarily equal one for each node.

Let P_α be a matrix consisting of Θ rows and λ columns; the value in the (i)th row and (j)th column will be $P_\alpha(i,j)$.

Let $P(i,j)$ equal $P_\alpha(i,j) - P_\beta(i,j)$, and, of course, let the matrix P equal $(P_\alpha - P_\beta)$. From all preceding definitions, the following points apply to the matrix P:

1) the sum of the elements in each row will equal zero;

2) each row will have at least two non-zero elements;

3) each column will have at least one non-zero element.

All future uses of the matrix P assume the above properties.

Examples

The following set of examples are presented here to illustrate the modelling methodology, and are later used to illustrate apportionment solutions. Numerical values are expressed in matrix form.

Example 1

This first example is a system consisting of three components and one unpriced resource. It is a trivial example, but serves to illustrate concepts. A diagram and the parameters of the system are shown in Figure 1.

The model represents a warehouse facility that distributes a product to two outlet facilities. Fifty percent of the warehouse's output over yearly time frame goes to each outlet. Average investment in each component equals 10,000 dollars. Node 1 is the abstract point at which the product is transferred from the warehouse to the outlets. The warehouse facility has no direct revenue, but creates 1,000 dollars worth of expenses per year. These expenses include facility depreciation, operating expenses, and the costs of purchasing the product. Outlet A produces 1,250 dollars and 250 dollars in revenue and expenses per year, respectively. Outlet B produces 750 dollars and 250 dollars in revenues and expenses per year, respectively. Contributions per year turn out to be -1,000 dollars, +1,000 dollars, and +500 dollars for the warehouse, outlet A, and outlet B, respectively.

Example 2

This example is a system consisting of three components and two unpriced resources. The model represents a two terminal motor carrier system. Component 1 represents the flow of fully ladened vehicles from the first to the second terminal; component 2 represents the flow of fully ladened vehicles carrying cargo in the opposite direction. Component 3 represents the flow of empty vehicles from the second back to the first terminal. The scenario is as follows.

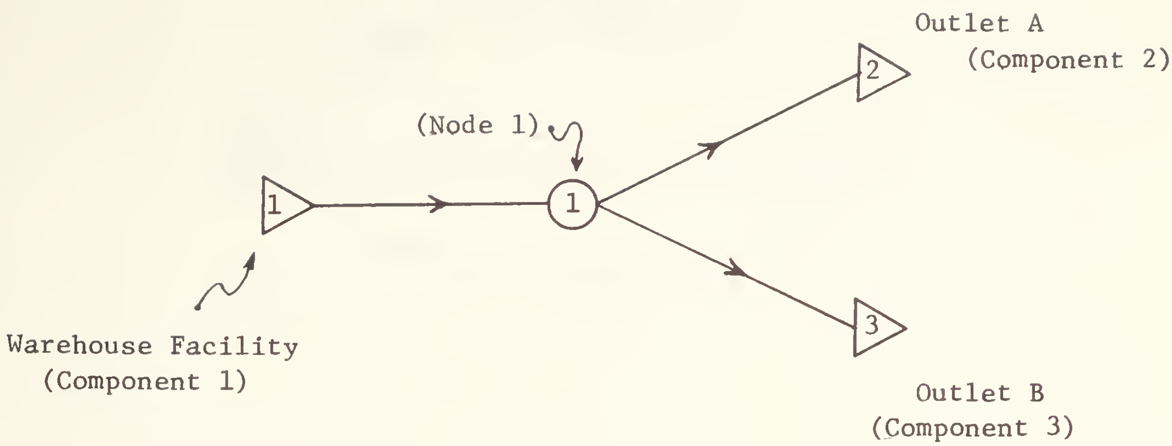
The two terminals are roughly one day driving time apart. There are 10 vehicles in the system, and the model represents a twenty operational day time frame (roughly one month, excluding weekends). The average value of each vehicle is 20,000 dollars. During a typical twenty day operation, 100 fully loaded vehicles carry cargo from terminal one to two. On the average, 90 percent of these vehicles return to terminal one fully loaded with other cargo, and 10 percent return empty. It is assumed for this example that there is no delay between unloading and loading, that loading and unloading times are negligible, and that the travel times between terminals are equal. Hence, 50 percent of vehicle investment is tied up in component 1, 45 percent in component 2, and 5 percent in component 3. Terminal facilities investments are not taken into account in this example.

Trucking expenses, which include vehicle depreciation, maintenance, driver and other operating expenses, are 100 dollars per vehicle movement between terminals. The expenses for moving an empty vehicle is assumed to be the same as for a loaded vehicle. The cost of loading and unloading a vehicle is 20 dollars. Revenues on cargo moving from terminal 1 to 2 are 220 dollars per vehicle; revenue on cargo moving in the opposite direction is also 220 dollars per loaded vehicle. The above data produces the parameters shown in Figure 2.

The unpriced resources in this example are the vehicles which are made available by incoming cargo and used by outgoing cargo at each terminal. Obviously, at each terminal the inflow of vehicles must equal the outflow of vehicles in the long run.

Example 3

This example is a system consisting of four components and two unpriced resources which are joint products. Component 1 produces two products, A and B. They are joint products in that the production of A necessarily produces B and vice versa. Component 2 is identical to component 1 except for the fact that their output ratios of A and B are different. Twenty percent of product A is produced by component 1; component 2 produces the remaining eighty percent. Component 1 produces ninety percent of product B, and the remaining ten percent is produced by component 2. Components 3 and 4 consume products A and B as raw materials, but in different ratios. Component 3 consumes fifty percent of A and forty percent of B; component 4 consumes fifty percent of A and sixty percent of B. Each of the four components have investment values of 1,000 dollars, and have expenses of 100 dollars per month. Components 3 and 4 each have monthly revenues of 200 dollars. The above structure and parameters are listed in Figure 3, using a monthly time frame.



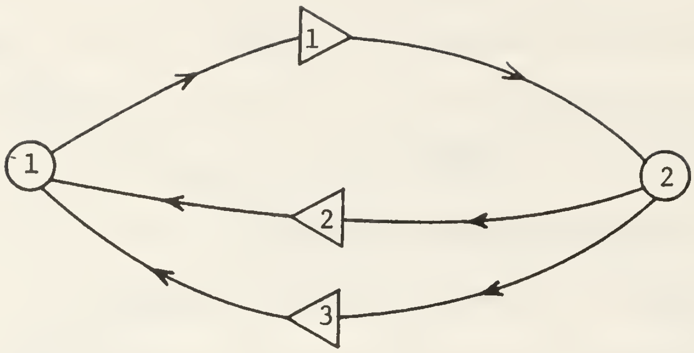
$$P_\alpha = \begin{bmatrix} 0 & .5 & .5 \end{bmatrix}$$

$$P_\beta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & .5 & .5 \end{bmatrix}$$

$$a = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \end{bmatrix} \quad r = \begin{bmatrix} 0 \\ 1250 \\ 750 \end{bmatrix} \quad e = \begin{bmatrix} 1000 \\ 250 \\ 250 \end{bmatrix} \quad c = \begin{bmatrix} -1000 \\ 1000 \\ 500 \end{bmatrix}$$

Figure 1, - Diagram and Parameters for Example 1



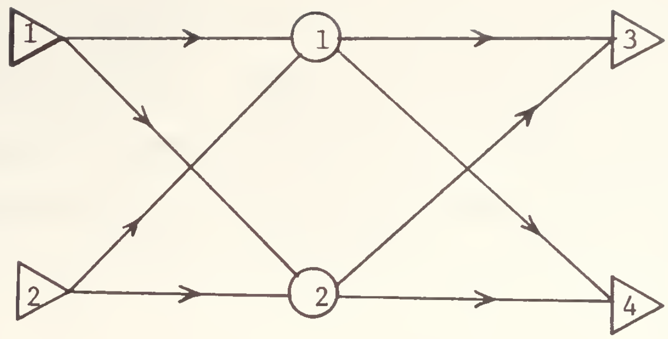
$$P_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .9 & .1 \end{bmatrix}$$

$$P_\beta = \begin{bmatrix} 0 & .9 & .1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -.9 & -.1 \\ -1 & .9 & .1 \end{bmatrix}$$

$$a = \begin{bmatrix} 100000 \\ 90000 \\ 10000 \end{bmatrix} \quad r = \begin{bmatrix} 22000 \\ 19800 \\ 0 \end{bmatrix} \quad e = \begin{bmatrix} 12000 \\ 10800 \\ 1000 \end{bmatrix} \quad c = \begin{bmatrix} 10000 \\ 9000 \\ -1000 \end{bmatrix}$$

Figure 2, - Diagram and Parameters for Example 2



$$P_\alpha = \begin{bmatrix} 0 & 0 & .5 & .5 \\ 0 & 0 & .4 & .6 \end{bmatrix}$$

$$P_\beta = \begin{bmatrix} .2 & .8 & 0 & 0 \\ .9 & .1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} -.2 & -.8 & .5 & .5 \\ -.9 & -.1 & .4 & .6 \end{bmatrix}$$

$$a = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix} \quad r = \begin{bmatrix} 0 \\ 0 \\ 200 \\ 200 \end{bmatrix} \quad e = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} \quad c = \begin{bmatrix} -100 \\ -100 \\ 100 \\ 100 \end{bmatrix}$$

Figure 3, - Diagram and Parameters for Example 3

Qualifications

Here, certain qualifications are made to clarify the ways in which components and nodes may be used in a model. For the most part, these qualifications do not restrict or limit the types of situations that can be modelled; rather, they identify the proper use of components and nodes when modelling.

Time Orientation

Inputs and outputs are measured in terms of the amounts of resources which flow into and out of nodes and components; flow is expressed as a rate (units per time period). The time period is typically equivalent to the time frame which the model represents; for example, if the model represents the monthly activities of a system, then inputs and outputs are expressed in units per month.

By the very nature of single time period models, operational fluctuations within a time period are not at issue; rather, the focus is on aggregate results. All the examples here are single time frame representations. Nevertheless, if fluctuations and timing effects had to be assessed, then multiple time frame models could be developed by conceptualizing components which carry resources from one time frame to the next.

Representation

The parameters and structure of the model should represent a complete operational cycle of the system being analyzed. That is to say, inputs, outputs, expenses and revenues must reflect a characteristic "snap-shot" of the system's operation.

For example, depreciation must be used to spread the cost of capital equipment over its useful life; it is typically irrational to expense the entire cost to the period in which it was purchased. Expense parameters should be an equitable portion of the historical and anticipated life cycle costs of the components.

Another example has to do with input and output flows. Flows must represent recurring operational activities. For instance, only the input resources which are regularly transformed into outputs should be used as the input flow parameter to a component. Start up inputs which build up inventories but never move through the components should not be included in flow parameters because they do not recur; rather, they should be considered as an investment expense.

The general point here is that periodic irregularities in a system must be smoothed so that resource flow, revenue, and expense parameters represent typical historical or anticipated operations.

Resources

Conceptually, each resource must be homogeneous in the sense that any one unit of the resource is equivalent to any other unit of the resource. For example, if the outputs of two different components are defined to be the same resource, then the outputs necessarily have identical attributes. The obvious implication here is that resources must be defined in a manner such that each group is homogeneous within itself.

Nodes

A node creates neither revenue nor expense. No losses of inputs or outputs occur when a node transfers them from one component to another. A node can well be thought of as a connector which only directs outputs of components towards inputs of other components; any expenses, revenues, or losses must occur in the modelled components.

A very important qualification regarding nodes and resources is that a resource can be represented by one and only one node. The reason for this is based on the premise that in order to fairly assess the components it is necessary to use one and only one transfer price for a particular resource. Due to the mathematical structure of the apportionment model here, this is accomplished by having one and only one node for each resource.

At this juncture, an important concept needs to be emphasized. Resources are differentiated by more than simple physical attributes. Attributes such as time and location also serve to distinguish resources from one another. For example, gadgets coming off a production line in New York are different resources than the gadgets coming off a production line in California because of the substantial distance between them; nevertheless, the same gadgets would be considered homogeneous in a common warehouse in Kansas City.

Components

If a component has more than one output, then these outputs must be joint products. That is to say, the production of any one output of a component must necessarily require the production of the other output(s). This qualification limits the number of conceptual activities that can be represented by a single component. For example, if activity 1 uses resource A to produce resource X, and activity 2 uses resource A to produce resource Y, then the above qualification implies that activity 1 should not be combined with activity 2 in the same component. Instead a different component should be used for each activity. In contrast, if an activity uses resource A to produce resource X, but cannot produce X without producing resource Y, then this activity is correctly represented as a single component.

The principle behind this qualification is that components should represent a unique and clearly definable resource transformation process so that apportionment results provide the greatest possible functional resolution.

Apportionment Concepts

Given a model of the firm, as described in previous sections, the focus here is on the criteria that are used to distribute revenues and expenses over the components which create and use unpriced resources. The apportionment concepts are initially presented in terms of two basic criteria; these criteria are sufficient to uniquely apportion revenues and expenses among components. This is followed by other criteria which are intrinsically, but not obviously, contained in the first two criteria. For the most part, a fair knowledge of matrix algebra and the optimization of quadratic forms will be necessary to understand the proofs.

Apportionment Notation

To simplify the ensuing discussions, some additional mathematical notation and definitions are provided here.

So far, most references to apportionment have used the terms revenues and expenses. This is an appropriate usage, however, in later discussions it will be much easier to use the term contribution instead. (Recall that a component's contribution is its revenues minus expenses.) In other words, the term apportioning contribution will be used instead of apportioning revenues and expenses; mathematically the two have equivalent meanings.

Let the contribution apportioned to the (j)th component be represented as the variable $y(j)$. In line with the discussion in the previous paragraph, $y(j)$ represents the revenue apportioned to component j minus the expenses apportioned to component j. Using the above definitions, let the λ by 1 column vector y represent apportioned component contributions.

The (j)th element in y will equal y(j).

Let the variable x(j) represent the observed contribution of component j minus the contribution apportioned to component j. This is mathematically represented as $x(j)=c(j)-y(j)$, or $y(j)+x(j)=c(j)$, or $y(j)=c(j)-x(j)$. In other words, x(j) represents the amount of actual contribution which is taken away from component j in order to arrive at apportioned contribution. Henceforth, x(j) will be referred to as the transferred contribution of component j. Let the λ by 1 column vector x represent transferred contributions of the components, where x(j) is the value of the (j)th element.

Using vector notation, the apportioned contribution vector plus the transferred contribution vector equals the actual contribution vector. Rearranging terms gives

$$y = c - x \qquad \text{(Eq. 1)}$$

In addition to the vector a, which has been previously defined to express component investment levels, the matrix A will also be used. It will be a square λ by λ diagonal matrix. The value of A(j,j) will be a(j); all non-diagonal elements will be zero.

The ratio of apportioned contribution to the investment level in each component will be used in later sections. Henceforth, this ratio will be referred to as the contribution ratio. The contribution ratio for the (j)th component is y(j)/a(j). The vector of contribution ratios can be represented as the inverse of the matrix A times y, or simply $A^{-1}\cdot y$. A contribution ratio can be viewed as the apportioned return on investment that accrues to a component during the time interval which the model represents.

The first criterion deals with the way in which contribution is to be transferred among components. The second criterion deals with how apportioned contribution is to be equitably balanced among components. Both are oriented around the inputs and outputs of unpriced resources. In essence, they specify two things: 1) any shifting of contribution among components must be done in a consistent manner using imputed transfer prices for unpriced resources; 2) the resulting shifts of contribution must produce balanced returns on investment throughout the system.

Nodal Orientations of Contribution Transfers

This criterion is expressed mathematically as follows: the values in x must be such that there exists a Θ by 1 column vector v which satisfies

$$x = P' \cdot v \quad . \qquad \text{(Eq. 2)}$$

Substituting $(P_\alpha - P_\beta)$ for P gives

$$x = P_\alpha' \cdot v - P_\beta' \cdot v \quad . \qquad \text{(Eq. 3)}$$

The above expressions are mathematically saying that the transferred contributions vector x must be representable as some vector v times the transpose of the known matrix P. In conceptual terms, the above expressions are saying that there should be a value for each node which expresses total contributions transferred through the node, and from which individual component transferred contributions can be derived by examining the flows of resources.

Specifically for component j,

$$x(j) = \left(\sum_{i=1}^{\Theta} P_\alpha(i,j) \cdot v(i) \right) - \left(\sum_{i=1}^{\Theta} P_\beta(i,j) \cdot v(i) \right), \qquad \text{(Eq. 4)}$$

where $v(i)$ represents the (i)th element in the vector v, and expresses the total contribution transferred through node i. Recalling that $P_\alpha(i,j)$ represents the proportion of the resources flowing through node i that are used by component j, it should be clear that the first summation in Equation 4 is the transferred contribution derived from all unpriced resources <u>used</u> by component j. Recalling that $P_\beta(i,j)$ represents the proportion of the resources flowing through node i that are provided by component j, it should be clear that the second summation is the transferred contribution derived from all unpriced resources provided to other components by j. All together, Equation 4 is saying that transferred contribution of component j must equal the transferred contribution of inputs minus the transferred contribution of outputs, whatever these values may be.

In terms of apportioned and actual contribution, the above implies that there must exist a v such that

$$y = c - P' \cdot v \ . \qquad \text{(Eq. 5)}$$

In terms of Equation 4, this is saying that apportioned contribution of a component must equal actual contribution <u>minus</u> the transferred contribution of inputs <u>plus</u> the transferred contribution of component outputs. In other words, transferred contributions represent the effects of imputed transfer prices for unpriced resources. It is important to note that this criterion does not in itself state how these transfer prices are to be ascertained, rather, it only specifies how transferred contributions are to be derived for them.

Another way of viewing this criterion is to say that transferred contributions must be consistent with one another. For example, if components 4 and 5 each provide fifty percent of an unpriced resource to component 6, then whatever the amount of contribution transferred from component 6 to 4 and 5, fifty percent must go to 4 and fifty percent must go to 5. The equations express this relationship.

In order to avoid a confusion that may arise later, it will be emphasized here that Equation 2 does not imply that the values in v will be unique. As will be shown later, it turns out that x will always be unique, but v may not be unique. The non-uniqueness of v does not weaken this criterion; as long as x is unique, the uniqueness of v is a moot point.

<u>Nodal Balance of Returns on Investment</u>

This criterion is mathematically stated as follows: apportioned contribution values in y must be such that

$$P \cdot A^{-1} \cdot y = \underline{0} \qquad \text{(Eq. 6)}$$

The above expression is mathematically saying that the P matrix times the inverse of the diagonal investment level matrix times the vector y must equal a Θ by 1 column vector of zeros. Substituting $(P_\alpha - P_\beta)$ for P, and rearranging terms gives:

$$P_\alpha \cdot A^{-1} \cdot y = P_\beta \cdot A^{-1} \cdot y \qquad \text{(Eq. 7)}$$

Conceptually, the above is saying that the weighted average of contribution ratios for the components providing resources to a node must equal the weighted average of contribution ratios for those components using resources from the node.

Specifically for node i,

$$\sum_{j=1}^{\lambda} P_\alpha(i,j) \cdot \frac{y(j)}{a(j)} = \sum_{j=1}^{\lambda} P_\beta(i,j) \cdot \frac{y(j)}{a(j)} \quad . \qquad \text{(Eq. 8)}$$

Recalling the definitions of P_α and P_β, it should be clear that the resource flows are used as weights for computing the weighted average of contribution ratios.

This criterion is essentially saying that the weighted average apportioned return on investment for the components on the inflow side of a node must equal the weighted average apportioned return on investment for the components on the out flow side of the node. In other words, imputed transfer prices must be such that averaged returns on investment are equalized on each side of each node. Support for this criterion comes from the argument that transfer prices must be such that the imputed effects do not favor producers of each unpriced resource over the consumers of each unpriced resource, and vice versa, because such favoritism would clearly bias the results.

Component's Viewpoint

These two basic criteria can be conceptualized in terms of a component's view of how revenues and expenses should be equitably distributed among components. The first criterion embodies the contention that transferred contributions must be based on standard imputed unit values for each unpriced resource. These unit values are uniformly applied to the components which produce and consume the resources. The point being that whatever one component "pays" or "receives" for one unit of an unpriced resource is equivalent to what another component "pays" or "receives" for that same resource; there is no discrimination. The second criterion embodies the general contention that the "imputed prices" for unpriced resources must affect the system so that the producing components aren't realizing greater, or lesser, average apportioned return on investment than the consuming components. From a systems standpoint, the producing and consuming components are depending on each other in regards to each unpriced resource, and therefore average contribution ratios of producing components should not be any more, or less, than consuming components for each resource.

Mathematical Solutions

It so happens that Equation 5 and Equation 6 are sufficient to determine unique values for apportioned contributions (y) and transferred contribution (x) for any known P,A, and c. For the moment however, it will be assumed that the Θ rows of P are linearly independent of one another. (Later it will be shown what needs to be done if the rows of P are not linearly independent; nevertheless, the solution values for y and x remain unaffected.)

Combining Equations 5 and 6 gives

$$\left(\begin{array}{c|c} I & P' \\ \hline P \cdot A^{-1} & 0 \end{array}\right) \cdot \left(\begin{array}{c} y \\ \hline v \end{array}\right) = \left(\begin{array}{c} c \\ \hline \underline{0} \end{array}\right) \qquad \text{(Eq. 9)}$$

The square partitioned matrix on the left has an inverse because its rows are linearly independent of one another; this comes from the assumption that the rows of P are independent.

Therefore,

$$\begin{pmatrix} y \\ \hline v \end{pmatrix} = \left(\begin{array}{c|c} I & P' \\ \hline P \cdot A^{-1} & 0 \end{array}\right)^{-1} \cdot \begin{pmatrix} c \\ \hline \underline{0} \end{pmatrix} . \qquad \text{(Eq. 10)}$$

Using inverse partitioning theorems, it follows that

$$v = \left(P \cdot A^{-1} \cdot P'\right)^{-1} \cdot P \cdot A^{-1} \cdot c . \qquad \text{(Eq. 11)}$$

The vector x can then be computed from Equation 2, and the vector y from Equation 5.

Dependence of Rows in P

It is shown in this section that if the rows in the matrix P are not all independent of one another, then there is still only one y which satisfies Equation 9. It is also shown how to find the solution values for x and y when the rows in P are dependent.

Consider two solution values for y in Equation 9. Let these two solutions be denoted as y_1 and y_2. It follows from Equation 5 that there would be a v_1 and v_2 such that

$$y_1 = c - P' \cdot v_1 \qquad \text{(Eq. 12)}$$

$$y_2 = c - P' \cdot v_2 \qquad \text{(Eq. 13)}$$

Substituting Equations 12 and 13 into Equation 6 gives Equation 14 and 15, respectively.

$$P \cdot A^{-1} \cdot (c - P' \cdot v_1) = \underline{0} \qquad \text{(Eq. 14)}$$

$$P \cdot A^{-1} \cdot (c - P' \cdot v_2) = \underline{0} \qquad \text{(Eq. 15)}$$

Subtracting Equation 15 from 14, cancelling common terms, and rearranging gives

$$P \cdot A^{-1} \cdot P' \cdot (v_1 - v_2) = \underline{0} \quad . \qquad \text{(Eq. 16)}$$

Multiplying Equation 16 by $(v_1-v_2)'$ gives

$$(v_1 - v_2)' \cdot P \cdot A^{-1} \cdot P' \cdot (v_1 - v_2) = 0 \qquad \text{(Eq. 17)}$$

Substituting $x_1 = P' \cdot v_1$ and $x_2 = P' \cdot v_2$ into Equation 17 gives

$$(x_1 - x_2)' \cdot A^{-1} \cdot (x_1 - x_2) = 0 \qquad \text{(Eq. 18)}$$

Now for Equation 18 to hold, (x_1-x_2) must equal $\underline{0}$ because all diagonal elements of the diagonal matrix A are positive. Therefore, it can be concluded that x_1 and x_2 are always equal, and also that y_1 and y_2 must always be equal, regardless of P.

It can be shown directly from the definition of vector independence that if the rows in the matrix P are not all independent of one another, then the v in Equation 9 is not unique. If $x = P' \cdot v$ and the columns of P' (rows of P) are not all independent, then there clearly exists different values of v which will give the same x.

It will now be shown how to calculate y and x when all the rows in P are not independent. First, remove the dependent row vectors from P; let this revised matrix be represented as P_*, and let Θ_* represent the number of rows in P_*. Using P_* instead of P in Equation 11 gives v_*, a Θ_* by 1 column vector. The value of x and y will then be

$$x = P_*' \cdot v_* \quad , \tag{Eq. 19}$$

$$y = c - P_*' \cdot v_* \quad . \tag{Eq. 20}$$

In other words, a solution is obtained by treating the dependent rows in P as if they didn't exist.

To confirm the above results, notice that the solution obtained in Equation 19 always satisfies the first criterion expressed in Equation 2. It follows directly from the definition of vector independence that if x is such that $x = P_*' \cdot v_*$, where P_* is composed of all the independent rows of P, then there exists a v such that $x = P' \cdot v$. Hence, the first criterion is met. Notice also that the y solution obtained in Equation 20 satisfies the second criterion expressed in Equation 6. Obviously, there exists a matrix B such that $P = B \cdot P_*$ because P_* is composed of all the independent rows of P. This means that Equation 6 can be expressed as

$$B \cdot P_* \cdot A^{-1} \cdot y = \underline{0} \tag{Eq. 21}$$

Now, the calculated y in Equation 20 obviously satisfies $P_* \cdot A^{-1} \cdot y = \underline{0}$ because P_* was used instead of P in the calculation. Therefore, Equation 21 will necessarily be satisfied because $B \cdot \underline{0} = \underline{0}$.

In summary, it has been shown in this section that the values of y and x are unaffected by the dependence in the rows of P. This was accomplished by first showing that only one y and one x meet the criteria expressed in Equations 5 and 6. It was then demonstrated that x and y can be validly determined by removing the dependent rows in P. Henceforth, the matrix P_* will be a Θ_* by λ matrix representing the Θ_* independent rows of P; the only restrictions on P are those stated in the section on modelling qualifications.

Additional Properties and Criteria

As mentioned previously, the first two criteria embody some properties which are not immediately obvious. This section discusses these properties and also some additional criteria that are met by the apportionment methodology.

Equality of Total Contribution and Apportioned Contribution

It is reasonable to require that the sum of the observed component contributions be equal to the sum of the apportioned component contributions. Mathematically this is saying that $\underline{1}' \cdot c$ equals $\underline{1}' \cdot y$, where $\underline{1}'$ is a row vector of ones. This already holds, and is demonstrated by multiplying $\underline{1}'$ times Equation 5.

$$\underline{1}' \cdot y = \underline{1}' \cdot c - \underline{1}' \cdot P' \cdot v \qquad \text{(Eq. 22)}$$

Now, from the qualifications on P stated in previous sections, the product of $\underline{1}' \cdot P'$ is always a vector of zeros; hence, $\underline{1}' \cdot y = \underline{1}' \cdot c$. Note that this is also saying that $\underline{1}' \cdot x = 0$ for any c.

Component Responses to Changes in Its Contribution

It is reasonable to require that if the observed contribution of a component is increased while others are held constant, then the resulting apportioned contribution of the component also increases. Mathematically this is saying that as $c(j)$ becomes greater while others are held constant, then the resulting solution value of $y(j)$ must become greater. This does occur, and is demonstrated by considering the following least squares problem. Suppose one was attempting to find $y(j)$ values which satisfy Equation 6, and also minimize ϕ where

$$\phi = \left(\frac{1}{2}\right) \cdot \sum_{j=1}^{\lambda} \left(c(j) - y(j)\right)^2 \cdot \left(1/a(j)\right) \qquad \text{(Eq. 23)}$$

The entire problem can be expressed in matrix form as follows: minimize ϕ, where

$$\phi = \left(\frac{1}{2}\right) \cdot (c-y)' \cdot A^{-1} \cdot (c-y) \qquad \text{(Eq. 24)}$$

and y is subject to the constraints

$$P \cdot A^{-1} \cdot y = \underline{0} \quad . \qquad \text{(Eq. 25)}$$

The P, A, and c are known as previously defined. The first order Lagrange conditions for the optimal solution to the problem expressed in Equations 24 and 25 are: the optimal y must be such that it satisfies Equations 25, and it must also be such that there exists a v which satisfies Equation 26.

$$y - c + P' \cdot v = \underline{0} \qquad \text{(Eq. 26)}$$

Notice that these conditions are the same as those expressed in Equation 9. Second order conditions for the minimization of ϕ are automatically met because A^{-1} is necessarily positive-definite.

Having shown that the y solution to the least-squares minimization problem expressed in Equations 24 and 25 is identical to the contribution apportionment solution, this least squares problem can now be used to show what happens to $y(j)$ when $c(j)$ is changed. After examining Equation 23, it is apparent that the $y(j)$ values are "pulled" toward $c(j)$ values through a weighted sum-of-the-squares penalty function. The $y(j)$ can respond to this pull because none of the elements in y are uniquely fixed by the $P \cdot A^{-1} \cdot y = \underline{0}$ constraint. Therefore, if a single $c(j)$ value is moved upwards, then the corresponding $y(j)$ value will also move upwards. Likewise, if a single $c(j)$ value is moved downwards, the corresponding $y(j)$ value will also move downwards.

It will now be shown that the magnitude of the change in $c(j)$ will always be greater than the magnitude of the change in $y(j)$. Consider another least-squares problem as follows: find values for x such that $x = P' \cdot v$, and ϕ is minimized, where

$$\phi = \left(\frac{1}{2}\right) \cdot \sum_{j=1}^{\lambda} \left(c(j) - x(j)\right)^2 \cdot \left(1/a(j)\right) \quad . \qquad \text{(Eq. 27)}$$

The entire problem can be expressed in matrix form as follows: minimize ϕ, where

$$\phi = \left(\frac{1}{2}\right) \cdot (c-x)' \cdot A^{-1} \cdot (c-x) \qquad \text{(Eq. 28)}$$

and x is subject to the constraint

$$x = P' \cdot v \quad . \qquad \text{(Eq. 29)}$$

The P, A, and c are known as previously defined. The first order Lagrange conditions for the optimal solution to the problem expressed in Equations 28 and 29 are: the optimal x must be such that there exists a v which satisfies Equation 29 and Equation 30.

$$P \cdot A^{-1}(c-P' \cdot v) = \underline{0} \qquad \text{(Eq. 30)}$$

A careful examination of these conditions reveal that they are another form of Equation 9. Second order conditions for the minimization of ϕ are automatically met because A^{-1} is necessarily positive-definite.

Having shown that the x solution to the least-squares minimization problem expressed in Equations 28 and 29 is identical to the contribution apportionment solution, the least-squares problem can now be used to show what happens to x(j) when c(j) is changed. It is apparent in Equation 28 and 29 that the x(j) values are "pulled" toward the c(j) values, and that the values in v will respond to this pull because there is at least one non-zero element in every column of P (row of P'). Therefore, if a single c(j) value is moved upwards, then the corresponding x(j) will also move upwards; likewise, x(j) moves down if c(j) moves down the scale.

Considering the above responses of both x(j) and y(j), along with the fact that c(j) = x(j) + y(j), it follows that an upward movement of c(j) is matched by an upward movement of <u>both</u> x(j) and y(j). This obviously implies that neither changes in x(j) nor changes in y(j) individually match the full change of c(j), rather, they both share in the change.

A precise summary of the above can now be stated through some additional notation. Let the λ by λ matrix X be derived from Equations 2 and 11 so that

$$x = X \cdot c \tag{Eq. 31}$$

where

$$X = P_*' \cdot (P_* \cdot A^{-1} \cdot P_*')^{-1} \cdot P_* \cdot A^{-1} \tag{Eq. 32}$$

Clearly, $x(j)$ increases by $X(j,j)$ when $c(j)$ is increased by one unit. The previous proofs have shown that for each j, $X(j,j)$ is greater than zero and less than one; in other words, the diagonal elements in X are positive, but less than one.

Let the λ by λ matrix Y be derived from Equations 1 and 31 so that

$$y = Y \cdot c \tag{Eq. 33}$$

where

$$Y = (I-X) \quad . \tag{Eq. 34}$$

It should be clear that the diagonal elements of Y are positive, but less than one.

Other approaches could be used to prove the above, as well as some of the following relationships; particularly useful would be theorems on symmetric matrices and positive-semidefinite matrices. Nevertheless, the approaches used here are sufficient to demonstrate the points, and provide useful insights to the problem.

As an aside, it should be noted here that the matrices X and Y are singular, and do not have inverses. This means that a number of different c vectors could produce the same y vector, and a number of different c vectors could produce the same x vector. Nevertheless, it also logically follows that two different c vectors cannot produce identical y vectors and identical x vectors simultaneously.

Cross Responses to Changes in Contribution

The previous section discussed how $y(j)$ and $x(j)$ respond to a change in $c(j)$; this section discusses how $y(j)$ and $x(j)$ respond to a change in $c(k)$, $j \neq k$. Clearly, $y(j)$ increases by $Y(j,k)$, and $x(j)$ increases by $X(j,k)$ when $c(k)$ is increased by one unit. It so happens that the $Y(j,k)$ and $X(j,k)$ values, $j \neq k$, can be either positive or negative, and their absolute values could possibly be greater than one. However, there are some general limitations which apply to these non-diagonal elements as discussed below.

It can be shown that the sum of the elements in any one column of X will equal zero, and the sum of the elements in any of column of Y will equal one. This roughly follows from the previously established fact that $\underline{1}' \cdot x$ equals zero for any c; this implies $\underline{1}' \cdot X \cdot c$ equals zero for any c, which in turn implies that $\underline{1}' \cdot X$ must equal a row vector of zeros, which shows the sum of the element in each column of x sum to zero.

It will now be shown that for all j and k that

$$0 \leq X(j,k)^2 < a(j)/a(k) \quad . \qquad \text{(Eq. 35)}$$

Consider the fact that $X' \cdot A^{-1} \cdot X$ equals $A^{-1} \cdot X$, which in turn implies that for all k

$$\sum_{j=1}^{\lambda} \left(X(j,k)^2/a(j) \right) = X(k,k)/a(k) \quad . \qquad \text{(Eq. 36)}$$

All the items in the summation are necessarily non-negative, therefore each item in the summation must be less than $X(k,k)/a(k)$. Having already shown that $X(k,k) < 1$, Equation 35 clearly holds. An evaluation of Equation 35 reveals that the magnitudes of the elements in X are limited by the square root of the ratios of capital levels; that is

$$\left| X(j,k) \right| < \sqrt{a(j)/a(k)} \quad . \qquad \text{(Eq. 37)}$$

This means that the potential response of $x(j)$ to a one unit change in $\dot{c}(k)$ can never be larger than $\pm\sqrt{a(j)/a(k)}$ regardless of P, and that the absolute magnitude of the response can only be larger than 1 when $a(j) > a(k)$. The same holds for y because $Y = I - X$.

An intuitive explanation for the situation where the absolute change in $\dot{y}(j)$ is larger than the change in $c(k)$, $j \neq k$, is the consideration that a contribution change in a component with a small capital investment level would require greater changes in apportioned contributions of those adjacent components which have large capital investments so as to maintain the contribution ratios as prescribed by Equation 6.

One final point needs to be made in this section with regards to the symmetry of responses to changes in $c(k)$. Consider $A^{-1} \cdot y$, which is the vector of contribution ratios. Using Equation 33, $A^{-1} \cdot y = A^{-1} \cdot Y \cdot c$. Notice that $A^{-1} \cdot Y$ is symmetric. This means that the response of

$y(j)/a(j)$ to a one unit change in $c(k)$ is equal to the response of $y(k)/a(k)$ to a one unit change in $c(j)$. Extending these results back to Y means that $Y(j,k)/a(j)$ equals $Y(k,j)/a(k)$, and $Y(j,k)/Y(k,j)$ equals $a(j)/a(k)$ provided that $Y(k,j) \neq 0$.

Adverse Cross Responses

This section discusses the situations where an increased contribution in one component results in a decrease in the apportioned contribution of another component. Such cases arise when the inputs of two components are connected to common nodes, and the ratios of consumed resources are roughly the same; the same holds for components with common outputs whose output resource ratios are the same. In a sense, this implies there is a competitive relationship between the two components in that they are performing similar functions within the system. An adverse response of apportioned contribution in one component to the increase of contribution in another component reflects this competitiveness.

To clarify this point, consider the following case. Suppose there are two different components j and k, and they perform identical functions within the system. Specifically, they are connected to the same nodes, and their resource flows are identical. This means mathematically that the (j)th column in P has the same coefficients as the (k)th column in P. For this section, let the vector p_j be the (j)th column, and the vector p_k be the (k)th column in P; as stated above $p_k = p_j$. Given A, P, and c, a solution for y and x can be found; it should be clear that $x(j) = p_j' \cdot v = p_k' \cdot v = x(k)$, and that $y(j) = c(j) - x(j)$, and $y(k) = c(k) - x(k)$. Now, consider what happens when $c(j)$ is made greater, $c(k)$ held constant. As shown previously, if $c(j)$ becomes greater, then $x(j)$ becomes greater, which means here that $y(k)$ becomes lesser because $x(k) = x(j)$. The intuitive explanation for this is that the two components are performing an identical function, and are, therefore, competing with one another in the performance of that function. Surely, component k should not be rewarded with additional apportioned contribution when component j becomes, say, more efficient by reducing expenses. Rather, component k should be penalized for not achieving the same reduction; as shown, the apportionment methodology justifiably penalizes component k in this situation.

Minimization of Apportioned Contribution Magnitudes

This section provides a verbal interpretation to the minimization problems expressed in Equations 28 and 29. The problem is mathematically equivalent to the minimization of $y' \cdot A^{-1} \cdot y$, subject to the constraints $y = c - x$, and $x = P' \cdot v$. In words, this problem is looking for "transfer prices" of unpriced resources which minimize the magnitudes of apportioned contributions in a weighted least-squares manner. Minimizing $y' \cdot A^{-1} \cdot y$ is attempting to equalize apportioned contributions around zero, with emphasis placed upon the components with the smaller investment levels. The importance of this, and the following section, is that they represent alternative, yet equivalent, mathematical viewpoints of apportionment concepts.

Minimization of Transferred Contribution Magnitudes

The minimization problem expressed in Equations 24 and 25 is mathematically equivalent to minimizing $x' \cdot A^{-1} \cdot x$, subject to the constraints $x = c - y$, and $P \cdot A^{-1} \cdot y = \underline{0}$. The objective of this problem is to find apportioned contributions which balance returns on investment around nodes while simultaneously minimizing the magnitudes of transferred contributions in a weighted least-squares manner. There is more "pressure" on components with small investment levels to have small transferred contributions than the components with large investment levels. Again, this is an alternative and equivalent view of the apportionment methodology.

Response to Investment Levels

It is reasonable to require that as the investment level in a component is increased, while other things are held constant, that the component's share of apportioned contribution also increase. The reasoning behind this requirement follows from the basic objective of the apportionment methodology which is to assess segmental returns on investment by distributing revenues and expenses over the components. Surely it is not reasonable to assign greater contribution (revenues minus expenses) to those components with the lesser investment levels, other things being equal; doing so would imply that the components with the smaller investments are responsible for the greater share of revenues and expenses. A key point to remember here is that the apportionment methodology is not necessarily attempting to measure the individual economic efficiency of each component, rather, it is measuring how each component logically shares in revenues and expenses created by a group of components so as to produce balanced returns on investment. Therefore, the greater shares of revenues and burdens of expenses should be applied to those components which account for the greater portion of capital investment.

From a mathematical standpoint, the above is saying that if a $y(j)$ is positive, then it must become more positive when $a(j)$ is increased, others held constant, because the component must share a larger portion of the positive returns. If $y(j)$ is negative, then it must become more negative when $a(j)$ is increased because the component must share a larger portion of the negative returns, or losses. Finally, if $y(j)$ is zero (neither positive nor negative), then a change in $a(j)$ will have no affect; this necessarily follows from the two previous statements.

It can be demonstrated that the above does hold by examining the least-squares problems expressed in Equations 28 and 29. Notice if $a(j)$ is increased, then the effect is that <u>less</u> emphasis is placed upon the squared difference between $c(j)$ and $x(j)$ in the equation being minimized. Therefore, if $(c(j) - x(j))$ is not equal to zero, then $x(j)$ will move further away from $c(j)$ when $a(j)$ is increased, which, in other words, is saying that the absolute value of $(c(j) - x(j))$ increases. Now, $y(j)$ equals $(c(j) - x(j))$ by definition, and it therefore follows that the absolute value of $y(j)$ increases as $a(j)$ increases when $y(j) \neq 0$. If $(c(j) - x(j))$ equals zero, then a change in $a(j)$ will have no affect on the least-squares problem, and $y(j)$ will remain at zero.

Effects of Consolidation

This section discusses the effects of combining one component with another component; it will be shown that in certain situations the effects will be very simple. Two cases are discussed. The first is where two components with equal contribution ratios are consolidated, and the second is where two proportional components are combined. In both cases, the resulting apportioned contribution for the composite component is the sum of the initial apportioned contributions.

From a mathematical standpoint, two components, say j and k, are consolidated by taking the elements in the (k)th column of P and adding them to the elements in the (j)th column of P; the result becomes a new column in P for the composite component, and the old j and k columns are eliminated. The investment level in the composite component is a(j) + a(k). The end result reflects the situation as if the old j and k components were modelled as a single component. The concern here is with how the solution changes when this is done.

For notational purposes v, y(j), y(k), c(j), c(k), a(j), and a(k) will denote values before the consolidation takes place, and the column vectors p_j and p_k are the old j and k columns of P, respectively.

The following mathematical generalizations are made about consolidation in order to simplify the concepts which follow later. The constraints imposed by the first basic criterion in Equation 5 imply that the apportioned contribution of the composite component can equal (y(j) + y(k)) without affecting v. This follows from the fact that (y(j) + y(k)) necessarily equals $c(j) + c(k) - (p_j' + p_k') \cdot v$ without changing the values in v. In other words, the first criterion implies that if the apportioned contribution in the composite component is y(j) + y(k), then the apportioned contributions in other components need not change to accommodate the new model structure.

The constraints imposed by the second basic criterion in Equation 6 imply that as long as Equation 38 holds, it is not necessary to revise the apportioned contributions of other components in the system. (The value of ϕ in Equation 38 is the apportioned contribution for the composite component.)

$$p_j \cdot (y(j)/a(j)) + p_k \cdot (y(k)/a(k)) = (p_j + p_k) \cdot \phi / (a(j)+a(k)) \qquad \text{(Eq. 38)}$$

This equation was derived from the observation that if the sum of the old contribution ratios times their respective P elements (the left side of Equation 38) equals the composite component's contribution ratio times its composite P elements (the right side of Equation 38), then Equation 6 will hold for the new model structure without changes in apportioned contributions of other components.

It can be shown that if component j is consolidated with component k, and $y(j)/a(j) = y(k)/a(k)$, then the resulting apportioned contribution for the composite component will be $(y(j)+y(k))$, and all other apportioned contributions will remain the same. As discussed above, the first basic criterion allows this, and for the second criterion it can be demonstrated that Equation 38 holds when both $\phi = (y(j)+y(k))$, and $y(j)/a(j) = y(k)/a(k)$.

It can also be shown that if component j is consolidated with component k, and $p_k = p_j \cdot (a(k)/a(j))$, then the resulting apportioned contribution for the composite component will be $(y(j)+y(k))$; other apportioned contributions remain the same as before. Again, the first basic criterion allows this, and for the second basic criterion it can be demonstrated that Equation 38 holds when $\phi = (y(j)+y(k))$, and $p_k = p_j \cdot (a(k)/a(j))$. (This latter mathematical statement is saying that the two components consume and produce identical resources in proportion to their investment level ratios.)

In summary, this section shows that when "similar" components are consolidated, the apportionment solution is not essentially changed. The term similar means here that either the components initially have equal contribution ratios, or their common resource flows are proportional to their investment level ratios. Although not formally addressed, it should be apparent that the consolidation of components which do not meet the above specifications would typically perturb the initial solution.

Internal Consistency

As an elementary point, it is important to realize that if actual contributions already meet the second basic criterion requiring nodal balance of returns on investment, then apportioned contributions will equal actual contributions. Mathematically this is saying that if the values in c are such that $P \cdot A^{-1} \cdot c = \underline{0}$, then $y = c$, and $x = \underline{0}$. This is demonstrated by considering the fact that there is one and only one y solution to Equation 9 for any c; therefore, if $P \cdot A^{-1} \cdot = \underline{0}$, then $y = c$, and $v = \underline{0}$. This is reasonable because if actual contributions already meet apportionment objectives, then surely apportioned contribution shouldn't be different.

A related fact here is that the y which results from any c will not change when multiplied by Y. That is, $y = Y \cdot y$. This follows from the preceding paragraph, and points out that $Y \cdot Y = Y$.

Another elementary point is that all the elements in y will equal zero only when c is a linear combination of the rows in P. Mathematically, $y = \underline{0}$ only if there exists a v such that $c = P' \cdot v$. This is demonstrated by again considering the unique y solution to Equation 9, and that when $c = P' \cdot v$, it necessarily forces y to equal zero. Conversely, it also follows that if $c \neq P' \cdot v$ for al possible v, then some elements in y cannot be equal to zero.

Effects of Uniform Capital Cost Rates

Depending on the nature of the managerial problem for which the apportionment methodology is being used, component expenses may or may not include opportunity costs of invested capital. This section will show the simple affect on apportioned contributions when it is included in a uniform manner.

Assuming that the opportunity cost rates of invested capital are equal in all components, the opportunity cost of capital for all components can be expressed as the vector $a \cdot \phi$, where a is the vector of investment levels as previously defined, and ϕ in this section is the opportunity interest rate for the time interval which the model represents. For example, if the interest rate was five percent, and three hundred dollars was the average investment in a component, then the opportunity cost of capital for the component would be $(300)\cdot(.05)$ or fifteen dollars.

Letting y_* and y represent apportioned contributions with and without capital costs, respectively, $y = Y \cdot c$ and $y_* = Y \cdot (c-a\cdot\phi)$. It obviously follows that $(y-y_*) = Y \cdot a \cdot \phi$, and, from the definition of Y, that $Y \cdot a \cdot \phi = (I-X) \cdot a \cdot \phi = a \cdot \phi - X \cdot a \cdot \phi$. Now, notice that $A^{-1} \cdot a \cdot \phi = \underline{1} \cdot \phi$, and from the structure of P that $P \cdot \underline{1} = \underline{0}$, and from Equation 32 that $X \cdot a \cdot \phi = \underline{0}$. Therefore, $(y-y_*) = a \cdot \phi$.

In summary, it has been shown here that uniform capital cost rates are reflected in apportioned contributions proportionally to investment levels in components. In the general case, however, where capital cost rates might not be equal in all components, it would be necessary to calculate individual effects.

Scaling Effects

It is important to note that using different scales, as long as they are linear, for the observed variables will not distort the result. That is to say, the results will be proportional regardless of the linear scale used for input parameters. For example, if the vector c is in terms of dollars instead of hundreds of dollars, then x and y will simply be a hundred times larger.

In addition, it is important to note that the resulting values of y and x for a given c are completely independent of the linear scale used for a, the investment levels vector. For example, if the vector c is in terms of dollars, then y and x will be in terms of dollars, and these dollar values will remain the same regardless of whether the vector a is in terms of dollars or thousands of dollars. The contribution ratios, on the other hand, would obviously be affected.

Furthermore, it is important to note that any row of P can be multiplied by any non-zero scalar without affecting y, x, or $A^{-1} \cdot y$. This occurs for the same reason as the preceding relationships, which is because such factors cancel out one another in the calculation of X in Equation 32. This means, among other things, that the elements of P can be expressed in terms of original resource flow values, if desired. Using proportions, as adopted in earlier sections, was done only for explanation purposes.

The properties discussed in this section are important because of practical considerations when finding numerical solutions to a problem. Re-scaling may be necessary in order to avoid unacceptable round-off errors in the computer, or manual calculations, particularly when solving large problems.

Apportionment and Optimization

Previous sections have discussed the apportionment methodology in terms of modelling, and distributing revenues and expenses over a system of components. The thrust of this section will be on the relationship between the apportionment methodology and system optimization. It so happens that, under some specific assumptions, the apportionment methodology produces results that have optimization overtones. The specific assumptions concern the response of contribution and resource flows to changes in component activity levels.

The purpose of this section is to highlight the fact that the mathematical aspects of the apportionment concept are applicable to some system optimization approaches. There is no intention of demonstrating that these aspects are efficient or necessarily appropriate; rather, the focus is on the general existence of these optimization aspects. In line with these objectives, some statements are made without rigorous proofs, particularly in the last part on general optimization.

The Perfectly Linear Model

For the moment, assume for each component that if the activity level increases by a small percentage, then revenues, expenses, unpriced resource inputs and outputs will each increase by that same percentage; similarly for a decrease. In effect, this assumption implies that revenue, expense, inputs and outputs are equal to known constants times the activity levels of the component. The principal point assumed here is that if a component's activity increases (decreases) by, say, one percent, then its revenues, expenses, inputs,and outputs each increase (decrease) by one percent.

With this assumption, the matrix P can now be used to describe balanced incremental changes of activity levels in the system. Let the λ by 1 column vector z represent incremental proportional changes in the current activity levels of the components, and let balanced values of z be those which satisfy the following equation.

$$P \cdot z = \underline{0} \quad \text{(Eq. 39)}$$

In words, this is saying that z is balanced if and only if the increase (decrease) in the production of each unpriced resource is exactly matched by the increase (decrease) in the consumption of each unpriced resource. A few trivial examples of balanced z values are: 1) a vector of zeros, which represents no change in the system; 2) a vector of ones, which represents a 100 percent <u>increase</u>, or doubling of current levels; 3) a vector of $(-.2)$ elements, representing a 20 percent reduction in activity levels. (There is <u>nothing</u> in Equation 39 saying that all elements in a balanced z must be equal or of the same sign; as stated, the above examples are trivial, and are used only to illustrate what is meant by a balanced z.)

For later use, notice that $c' \cdot z$ equals the change in total contribution that occurs with a change of z in the system under the perfectly linear assumption. A positive value of $c' \cdot z$ is an increase, and a negative value is a decrease in total contribution.

Contribution Ratios and Total Contribution Improvement

Now let's consider contribution ratios as a value for z; that is, let $z_* = A^{-1} \cdot y$ where y is a solution for known P, A, and c.

Firstly, notice that z_* will always be balanced because $P \cdot A^{-1} \cdot y$ necessarily equals $\underline{0}$. Secondly, as will be shown in just a moment, the quantity $c' \cdot z_*$ will always equal zero when $y = \underline{0}$, and, more importantly, will always be greater than zero when $y \neq \underline{0}$. Furthermore, if $c' \cdot z_*$ equals zero, then there will exist no other z which satisfies Equation 39 and simultaneously produces $c' \cdot z > 0$.

In verbal terms, the above results imply that if z_* is not equal to $\underline{0}$, and is used as an incremental proportional change in current component activity levels, then the new activity levels would be balanced, and would produce greater total contributions. If z_* is equal to $\underline{0}$, then the implication is that there exists no balanced z which will produce greater total contributions.

The value of these implications is that under the linear assumption, the magnitudes and signs of contribution ratios reflect the degree to which the activities in each component could be increased or decreased, in a balanced manner, so as to produce greater total contributions; furthermore, contribution ratios would be indicative of when it is impossible to improve total contribution. Additional aspects of system optimization are covered later, but now for the proofs.

Consider the following maximization problem: find a vector y which satisfies Equation 41, and maximizes ϕ, where

$$\phi = c' \cdot A^{-1} \cdot y - y' \cdot A^{-1} \cdot y \cdot \left(\frac{1}{2}\right), \qquad \text{(Eq. 40)}$$

subject to

$$P \cdot A^{-1} \cdot y = \underline{0} \quad . \qquad \text{(Eq. 41)}$$

(P, A, and c are as previously defined.)

The Lagrange condition for the maximizing solution is a y which satisfies Equation 41 and, along with v, satisfies Equation 42.

$$A^{-1} \cdot c - A^{-1} \cdot y - A^{-1} \cdot P' \cdot v = \underline{0} \qquad \text{(Eq. 42)}$$

Notice that by multiplying Equation 42 by the matrix A will give an equation which together with Equation 41 are identical to the conditions in Equation 9. Therefore, the maximization solution in the problem expressed in Equations 40 and 41 can be used to characterize the apportionment solution of Equation 9.

Now, notice that the maximized value of ϕ in Equation 40 will always be non-negative. This follows from the contradictory results which arise if it is assumed that the maximized value of ϕ could be negative; consider that $y = \underline{0}$ is a solution that satisfies Equation 41, and produces ϕ equal to zero, which is greater than a negative ϕ; obviously, the negative ϕ could not be a maximum. Having shown that maximum ϕ is always greater than or equal to zero, it logically follows that $c' \cdot A^{-1} \cdot y$ must always be greater than or equal to $y' \cdot A^{-1} \cdot y \cdot \left(\frac{1}{2}\right)$. Notice $y' \cdot A^{-1} \cdot y$ is always positive when $y \neq \underline{0}$, and zero when $y = \underline{0}$. It now follows that $c' \cdot A^{-1} \cdot y$ must be greater than zero when $y \neq \underline{0}$, and that the only way that $c' \cdot A^{-1} \cdot y$ can equal zero is when $y = \underline{0}$.

In previous sections it was shown that $y = \underline{0}$ only when c is such that there exists v where $c = P' \cdot v$; hence, $c' \cdot A^{-1} \cdot y$ equals zero only when $c = P' \cdot v$. Also consider that if $c = P' \cdot v$, then for every z which satisfies $P \cdot z = \underline{0}$ will also satisfy $c' \cdot z = 0$; this follows from the observation that $c' \cdot z = v' \cdot P \cdot z$, and therefore, if

$P \cdot z = \underline{0}$, then $v' \cdot P \cdot z$ clearly equals zero. Synthesizing the conclusions of this paragraph show that when $c' \cdot A^{-1} \cdot y = 0$, no other balanced z produces improved total contributions under the perfect linearity assumption.

It is interesting to note that the optimal value of ϕ in Equation 40 will equal the optimal value of ϕ in Equation 28. This can be shown from the fact that $A^{-1} \cdot Y = Y' \cdot A^{-1} \cdot Y$, and therefore, $c' \cdot A^{-1} \cdot y$ always equals $y' \cdot A^{-1} \cdot y$. The value of ϕ in Equation 28 equals $\left(\frac{1}{2}\right) \cdot y' \cdot A^{-1} \cdot y$ as does the value of ϕ in Equation 40.

<u>Contribution Ratios as a Maximizing Contribution Step</u>

Retaining the linear assumptions, suppose that there is a constraint restricting changes in component investment levels of the form

$$\left(\frac{1}{2}\right) \cdot z' \cdot A \cdot z \leq \beta \quad , \qquad \text{(Eq. 43)}$$

where: A is the investment level matrix, as previously defined; z is the vector of incremental proportional changes in activity levels, as previously defined; and β is a known small positive constant. Given A and β, Equation 43 limits proportional changes to an ellipsoidal region around the origin; those components with small investment levels can change more freely than large components. The idea being, for example, that a one percent change in a large component produces a greater "shock" to the system than a one percent change in a small component.

Now, let's find a balanced z which maximizes $c' \cdot z$, and meets the constraint imposed by Equation 43; in other words, find an incremental change, or step, which creates the greatest improvement in total contribution while meeting both the balanced z requirement (Equation 39), and the change limitation (Equation 43). Although not proven here, the optimal solution to z is always a multiple of $A^{-1} \cdot y$, where y is the apportioned contribution solution in Equation 9.

Again, the point is that the magnitudes and signs of contribution ratios, calculated from apportioned contributions, reflect the degree to which the activity levels could be changed in a stepwise manner to optimally increase total contribution under the scenario stated herein.

The General Optimization Case

In the general case, the linear assumptions do not hold, and therefore marginal and incremental effects must be evaluated in order to optimize a system. In this section, a marginal optimization approach is discussed which relates to the mathematical structure of contribution apportionment.

Consider the following case. Let the matrix P be composed of elements which characterize marginal resource effects; specifically, let each $P(i,j)$ equal the incremental affect on the flow of unpriced resources through the (i)th node for a small proportional change in the activity level of the (j)th component. Let the vector c be composed of the elements which characterize marginal contribution effects; specifically, let $c(j)$ equal the incremental affect on total contribution of a small proportional change in the activity level of the (j)th component. Let the vector z represent possible changes in activity levels as before. Balanced z must satisfy $P \cdot z = \underline{0}$, and will produce a change in total contribution of approximately $c' \cdot z$ where, of course, the magnitude of the elements in z are small. Let the constraint in

Equation 43 be used to keep the magnitudes of the elements in z small, where A is the investment level matrix as before, and β is a small positive value.

Now, a balanced value for z which maximizes $c' \cdot z$, while meeting the constraint of Equation 43, would produce the same or higher levels of total contribution when applied to the system. At the new operating level, the values of P, c, and A would be re-evaluated, a new optimal z would be found, and the process repeated, presumably realizing a significant improvement in total contribution during each step. Whether or not this procedure would eventually reach a global optimum depends on a number of factors including the general nature of revenue, expense, input and output functions; also a question of convergence arises. Nevertheless, if the system were at the total contribution maximizing level, then the optimal z of the problem posed above would more or less equal a vector of zeros. In other words, it is a necessary condition for contribution maximization that the optimal z does not suggest that a substantial movement away from the current operating level will produce higher profits.

The optimal z value at any one stage, as you may have already guessed, is equal to $A^{-1} \cdot y$, where y is the solution to Equation 9 with the above definitions applying to P, A, and c. It should be apparent that this is not an easy computational method for contribution maximization, but it serves to point out the fact that the mathematical nature of the apportionment methodology is compatible with the incremental aspects of optimization.

Numerical Examples

In this section, contributions are apportioned for the examples previously presented in the section on modelling. Figures 1 through 3 displayed the diagrams and parameters of those examples. They are covered here in the same order, and Tables 1 through 3 display the respective apportionment results. Numerical results are shown as exact fractions, along with rounded decimal representations.

It must be emphasized here that these examples are trivial, and therefore the apportionment results will yield rather obvious conclusions. In fact, the purpose of these examples is to demonstrate that the methodology gives reasonable results. The reader should not be looking for dramatic revelations. As previously mentioned, the practical value of the apportionment methodology is primarily realized in the analysis of complex systems.

Example 1

Table 1 shows the calculated values for the Y and the $A^{-1} \cdot Y$ matrices derived from the P and A matrices; in addition, the x and y vectors resulting from c are shown along with the value of $v(1)$.

These results show that the total imputed transfer price for the unpriced resource is roughly 1166 dollars; deducting expenses from this gives an apportioned contribution of roughly 166 dollars for component 1. Splitting the 1166 in half (because each outlet shares equally in the output of the warehouse), gives 583 dollars which when deducted from each outlet's contribution produces apportioned contributions of +417 dollars and -83 dollars for components 2 and 3, respectively. Taking investment levels into account gives contribution ratios of 1.66 percent, 4.17 percent, and -.83 percent for components 1, 2, and 3, respectively. The weighted average of these returns is, of course, 1.66 percent; this reflects the fact that total contribution is 500 dollars for the 30,000 dollar investment.

Interpretation

These results are certainly consistent with "common sense." The example could have been examined through a simple cost analysis by equally allocating the 1,000 dollar expense of component 1 to components 2 and 3; such an approach shows that outlet A has a "profit" of 500 dollars and outlet B breaks even. This provides a meaningful contrast with apportionment results as discussed in the paragraph below.

The simple cost analysis suggests that outlet B is breaking even whereas apportionment results suggest outlet B has negative returns. Both approaches show outlet B in an unfavorable light, however, the apportionment approach amplifies the situation; the negative result exists because the apportionment technique inherently distinguishes the profitable performance of outlet A from the breakeven performance of B, and thus implies an opportunity loss associated with the resources tied up in outlet B.

An examination of the coefficients in the Y matrix provides some additional insights to this example. Recall that $Y(j,k)$ is the element in the (j)th row and (k)th column of Y, and that it represents the amount that $y(j)$ changes for a one unit change in $c(k)$. The first column of Y shows that the expenses of the warehouse are equally distributed among all three components. The second column of Y shows that for each dollar of contribution created in outlet A, (1/3) is fed back to component 1, (5/6) remains in component 2, and component 3 has its apportioned contribution reduced by (1/6). This latter negative effect arises because the two outlets are in a relative competition with one another; surely, the apportioned contribution of outlet B should not improve when the competing outlet A increases its, say, efficiency. The third column of Y is a reflection of the second column for outlet B.

An Alternative Derivation

It is worth while to point out here that the coefficients in the Y matrix for this, and similar, examples could be derived somewhat independent of the basic apportionment criteria. This alternative approach provides a different vantage point from which to view apportionment results; the approach is discussed in the following paragraphs.

For the moment, consider the problem of distributing the contributions among the three components in the example without reference to basic apportionment criteria. In other words, derive values for the coefficients in Y from a reasonable set of assumptions which are justified independent of the basic apportionment criteria. This is accomplished by using the following assumptions and logic.

Firstly, assume that the $A^{-1} \cdot Y$ matrix must be symmetric. This is based on the intuitive reasoning that changes in contributions should have equivalent reciprocal effects on contribution ratios. That is saying, for example, that a one dollar change in $c(1)$ should affect $y(2)/a(2)$ exactly the same way as a one dollar change in $c(2)$ affects $y(1)/a(1)$.

Secondly, assume that the sum of actual contributions must equal the sum of apportioned contributions. This means that the sum of the coefficients in any column of Y must equal one.

Thirdly, assume that the contribution (expenses) of component 1 must be equally shared between components 2 and 3. This is justified from the fact that component 1 equally supports components 2 and 3.

Fourthly, assume that a one dollar increase in the contribution of component 1 along with a fifty cent decline in both the contribution of components 2 and 3 will not change apportioned contributions. The reasoning behind this assumption comes again from the fact that component 1 equally supports components 2 and 3, therefore, a dollar's worth of actual contribution realized in component 1 should be equivalent to the realization of fifty cents in both components 2 and 3. (The third assumption differs from the forth assumption in that the former specifies how the contribution of component 1 must be shared with components 2 and 3, whereas the latter assumption specifies how the contributions of 2 and 3 must relate to 1.)

Fifthly, assume that a one dollar increase in the contribution of component 2 and a one dollar decrease in the contribution of component 3 will result in a one dollar increase in the apportioned contribution of component 2 and a one dollar decrease in the apportioned contribution of component 3. This follows from the general reasoning that components 2 and 3 are performing similar functions with respect to component 1. Therefore, an increase in the contribution of component 2 matched by a corresponding decline in the contribution of component 3 should have no affect on the apportioned contribution of component 1, and furthermore, the shift in actual contribution from 2 to 3 should be reflected in apportioned contributions. (The fourth assumption differs from the fifth assumption in that the former specifies how the contributions of components 2 and 3 must relate to component 1, whereas the latter specifies how the contributions of components 2 and 3 relate to each other.)

With the above assumptions, the values of the coefficients in Y can now be deduced. The first assumption, along with the fact that the investment levels in each component are equal, means that Y itself must be symmetric. The third assumption implies $Y(2,1)$ must equal $Y(3,1)$. Combining this with the second assumption, all the elements in the Y matrix can be expressed in terms of the two unknowns α and β as follows:

$Y(2,1) = Y(3,1) = Y(1,2) = Y(1,3) = \alpha$; $Y(3,2) = Y(2,3) = \beta$; $Y(1,1) = 1-2\alpha$;

$Y(2,2) = 1-\alpha-\beta$; and $Y(3,3) = 1-\alpha-\beta$.

Applying the fourth assumption to the above produces the equation $\alpha = (.5)(1-\alpha-\beta) + (.5)(\beta)$, which when solved defines the value of α to be (1/3); β cancels out. Applying the fifth assumption to the above produces the equation $1 = (1-\alpha-\beta) - \beta$, which when solved gives $\beta = -(1/6)$.

In summary, this exercise shows that there is an alternative way of deriving the coefficients in the Y matrix for this example, and in doing so illustrates the characteristics of apportionment results; presumably, alternative approaches exist for any example.

Example 2

This transportation example is characteristic of most transportation system models in that there exists dependency among the rows in the P matrix. In this case, row two is the negative of row one and vice versa. As covered in the previous sections, Y is calculated by eliminating dependent rows in P until all remaining rows are independent; in this case, either row one or row two can be eliminated, leaving the other. Regardless of which one is eliminated, the calculated apportionment values are listed in Table 2.

The apportioned contributions, along with the contribution ratios, reveal component 2 in the most favorable light, followed by component 1. This is intuitively reasonable from the standpoint that the per vehicle cargo revenue is equal in components 1 and 2, but the cargo in component 1 is creating the necessity for transporting empty vehicles back to the first terminal; hence, there is a backhaul expense for which component 1 is responsible. Furthermore, a unit reduction of cargo traffic in component 2 (component 1 traffic held constant), would have worse economic consequences on the system than a unit reduction of cargo traffic in component 1 (component 2 traffic held constant); the implication is that the cargo traffic of component 2 is more valuable than that of 1. This is reflected in apportionment results.

Example 3

This example is interesting because the joint products make a conventional analysis nearly impossible with the limited amount of information given. (If the market values of the joint products A and B were known, then it would be easy to evaluate the components using conventional accounting techniques.) Nevertheless, the apportionment methodology provides some evaluations without additional information; the results are shown in Table 3.

An explanation of these results are based on observations about resource flows. Notice that equal amounts of resource A are used by components 3 and 4; also, notice that component 4 uses more of resource B than does component 3. Therefore, component 4 consumes more total resources than does component 3, which implies that component 4 should be responsible for the greater share of the resource production expenses. Coupling this with the fact that contributions of components 3 and 4 are equal means that component 3 should be shown in a more favorable light than 4; apportionment results show this.

Apportionment results also show component 1 in a less favorable light than component 2. This can be explained by the fact that component 1 produces the greater share of resource B, and that component 4 consumes the greater share of B. Having reasoned in the previous paragraph that component 4 is in a less favorable position than 3, it follows that component 1 is being penalized, in a minor way, for being the major source of product B.

Taken as a whole, the absolute differences between the apportioned contributions are small in magnitude when compared to the investment levels. The overall conclusions here is that the returns on 1 are a little less than the returns on 2, but that the differences between components 3 and 4 are more substantial.

Table 1, - Apportionment Results for Example 1

$$Y = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix} \cdot (1/6) \qquad A^{-1} \cdot Y = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix} \cdot (1/60000)$$

$$y = \begin{bmatrix} 500/3 \\ 1250/3 \\ -250/3 \end{bmatrix} \doteq \begin{bmatrix} 166 \\ 417 \\ -83 \end{bmatrix} \qquad A^{-1} \cdot y \doteq \begin{bmatrix} .0166 \\ .0417 \\ -.0083 \end{bmatrix}$$

$$v(1) = 3500/3 \doteq 1166 \qquad x = \begin{bmatrix} -3500/3 \\ +1750/3 \\ +1750/3 \end{bmatrix} \doteq \begin{bmatrix} -1166 \\ 583 \\ 583 \end{bmatrix}$$

$$c' = \begin{bmatrix} -1000 \\ 1000 \\ 500 \end{bmatrix} = x + y \doteq \begin{bmatrix} -1166 \\ 583 \\ 583 \end{bmatrix} + \begin{bmatrix} 166 \\ 417 \\ -83 \end{bmatrix}$$

Table 2, - Apportionment Results for Example 2

$$Y = \begin{bmatrix} 10 & 10 & 10 \\ 9 & 11 & -9 \\ 1 & -1 & 19 \end{bmatrix} \cdot (.05) \qquad A^{-1} \cdot Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 11/9 & -1 \\ 1 & -1 & 19 \end{bmatrix} \cdot (.000005)$$

$$y = \begin{bmatrix} 9000 \\ 9900 \\ -900 \end{bmatrix} \qquad A^{-1} \cdot y = \begin{bmatrix} .09 \\ .11 \\ -.09 \end{bmatrix}$$

$$x = \begin{bmatrix} 1000 \\ -900 \\ -100 \end{bmatrix}$$

$$c = \begin{bmatrix} 10000 \\ 9000 \\ -1000 \end{bmatrix} = x + y = \begin{bmatrix} 1000 \\ -900 \\ -100 \end{bmatrix} + \begin{bmatrix} 9000 \\ 9900 \\ -900 \end{bmatrix}$$

Table 3, - Apportionment Results for Example 3

$$Y = \begin{bmatrix} .2678 & .2410 & .1560 & .3380 \\ .2418 & .2558 & .3020 & .2040 \\ .1560 & .3020 & .7838 & -.2382 \\ .3380 & .2040 & -.2382 & .6998 \end{bmatrix} \cdot (1/1.0036)$$

$$y = \begin{bmatrix} -1.56 \\ .84 \\ 8.76 \\ -8.04 \end{bmatrix} \cdot (1/1.0036) \doteq \begin{bmatrix} -1.56 \\ .84 \\ 8.73 \\ -8.01 \end{bmatrix}$$

$$v = \begin{bmatrix} 116 \\ 84 \end{bmatrix} \cdot (1/1.0036) \doteq \begin{bmatrix} 116 \\ 84 \end{bmatrix}$$

$$x = \begin{bmatrix} -98.8 \\ -101.2 \\ 91.6 \\ 108.4 \end{bmatrix} \cdot (1/1.0036) \doteq \begin{bmatrix} -98.44 \\ -100.84 \\ 91.27 \\ 108.01 \end{bmatrix}$$

Applications

The apportionment concept can be used in a number of ways for managerial and administrative purposes. A few conceptual applications are discussed in the following sections.

Specific rules for determining the resources and components that should be modelled for a given situation will not be formulated here. Nevertheless, the following general remarks should be kept in mind.

The apportionment model presented here is limited to a simple case where internally transferred resources have either a precise external value, or a completely ambiguous external value. In situations where definite lower and upper bounds on the values of resources exist, the so called general model should be used.

In most applications it is necessary to develop several alternative model structures, and then select the one which is most congruent with the economic or management problem being analyzed. A place to start is the $P \cdot A^{-1} \cdot y = \underline{0}$ equations; each of these equations should be conceptually analyzed for consistency with the problem. Superfluous components and resources should be eliminated.

A below average, or negative, contribution ratio does not necessarily imply that a segment should be automatically eliminated from the enterprise. The correct implication is that the segment, and the function which it is performing, needs review. It may be that the segment cannot be eliminated because of its essential nature. Similarly, a high contribution ratio does not necessarily imply that investment should be intensified; it may be impossible to expand and proportionally increase contributions. The point here is that apportionment results identify problem areas; the determination of corrective action requires further analysis, presumably using additional information and methodologies.

Segmental Evaluation

The most obvious application is the assessment of segmental investment returns. In this application, contribution ratios would be compared to reveal the order in which the various activities share in apportioned returns. Initially, the appropriate managerial action would be to investigate, explain and understand any variations. Steps might then be taken for improvement by using, for example, marginal revenue and cost analyses.

When the apportioned returns in a component are either above or below normal, the apportionment methodology is saying that the component appears economically "out-of-kilter" with respect to the rest of the system. A below or above average return must be attributable to factors other than a component's producer or user function because apportioned contributions balance the returns among the producers and users of each resource. For example, an out of the ordinary contribution ratio of a producing component cannot be rationally attributed to the fact that the component is a producer, say as opposed to a marketing component, because the transfer prices which apportion revenues and expenses are such that the average contribution ratios for producers and users of each resource are equal.

Unfavorable apportioned returns in a component imply that the component itself, and/or the components to which it is linked, are not producing relatively adequate revenues to cover costs. Corrective actions might be any of the following:

1) not doing anything because of the functional necessity of the component;
2) changing operating procedures;
3) price changes;
4) or even elimination of the activity.

Segmental evaluation could be applied to most any corporate enterprise, however, it would produce the most enlightening results in areas where joint products with ambiguous values flow across departmental lines.

Performance Measures

Segmental evaluations in consecutive time frames can serve as a means of measuring component performance through time. Using contribution ratios to measure the performance of a component over time motivates department managers to take into account system interactions. The first numerical example, as described in Figure 1 and Table 1, illustrates this point as follows.

Suppose there is a packaging task that is currently being performed by the two outlets, but could be performed cheaper at the warehouse. Specifically, suppose it costs the two outlets 15 dollars each, whereas the warehouse could do both jobs for 27 dollars (a three dollar savings). An examination of the Y matrix reveals that if both tasks were shifted back to the warehouse, then apportioned contributions would increase by one dollar in each of the three components. The point here is that the implementation of the cost saving procedure would improve apportioned contributions in <u>all</u> components.

Contrast the above result with a situation where the three components are being evaluated in terms of expenses only; the cost saving procedure would be favored by the outlets because the procedure would lower their individual expenses, but would be unfavorable to the warehouse because the procedure would increase its expense. These "self-centered" conflicts turn attention away from the real issue of implementing plans which truly lower total costs.

In addition, recall that the apportionment methodology highlights components which are in competitive situations. Referring to the first numerical example again, the two outlet components have such a relationship. Reduced expenditures in outlet A have a negative impact on the apportioned contributions of outlet B, and vice versa. This produces a useful comparison of performance.

Pricing

Whereas the above two applications more or less look back on historical occurrences, this application deals with anticipating future operations so that prices of final products of an enterprise can be adjusted to produce equitable returns on investments among components. Although no formal justifications or detailed examples are given here, it should be conceptually apparent that the apportionment methodology can be used to identify the prices of final products which produce standardized contribution ratios throughout a system. This would be potentially useful in regulated industries where there exist joint products. The desirability of using the apportionment approach remains to be examined in light of contemporary welfare economic theory.

Summary

The apportionment methodology is oriented toward the evaluation of economic systems that have intermediate and joint products of ambiguous value which flow across segmental lines. It is a method for developing transfer prices which attempt to balance investment returns among the various segments of a system, and in so doing, imputes intermediate and joint product values. The results reveal the degree to which each component shares in system wide investment returns and/or losses. Under additional assumptions, it is also related to optimization techniques.

Applications include the following areas: segmental evaluation where the objective is to assess how each segment relatively shares in total investment returns; component performance measurement where the objective is to motivate component (department) managers to strive toward policies and operating procedures which benefit the system as a whole; and price setting where the objective is to standardize the apportioned returns among components for profit or regulatory motives.

FOOTNOTES

1. The term "allocated" is used here in the common economic and accounting sense that joint costs cannot be rationally allocated when the analyst is limited to cost data alone; see [21, p.165] or [17, Chapter 14]. It is certainly realized here that if additional data is available, such as production constraints and revenue schedules, then there are reasonable schemes which, so to speak, allocate joint and even overhead costs; see [22],[19],[23],[11], and [12].

2. See [6],[7],[14], and [9].

3. See [1],[20], and [2].

4. See [3] and [8].

5. The modelling of a system here is somewhat along the lines of an input-output model, see [4]; nevertheless, the approach and objectives here are substantially different, and the reader should be aware of this. Likewise, there is a similarity to reciprocal cost allocation models, see [10], but again the approach and the objectives here are different.

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